Is Relativistic Quantum Mechanics Compatible with Special Relativity?

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The transformation from a time-dependent random walk to quantum mechanics converts a modified Bessel function into an ordinary one together with a phase factor $e^{i\pi/2}$ for each time the electron flips both direction and handedness. Causality requires the argument to be greater than the order of the Bessel function. Assuming equal probabilities for jumps ± 1, the normalized modified Bessel function of an imaginary argument is the solution of the finite difference differential Schrödinger equation whereas the same function of a real argument satisfies the diffusion equation. In the nonrelativistic limit, the stability condition of the difference scheme contains the mass whereas in the ultrarelativistic limit only the velocity of light appears. Particle waves in the nonrelativistic limit become elastic waves in the ultrarelativistic limit with a phase shift in the frequency and wave number of $\pi/2$. The ordinary Bessel function satisfies a second order recurrence relation which is a finite difference differential wave equation, using non-nearest neighbors, whose solutions are the chirality components of a free-particle in the zero fermion mass limit. Reintroducing the mass by a phase transformation transforms the wave equation into the Klein-Gordon equation but does not admit a solution in terms of ordinary Bessel functions. However, a sign change of the mass term permits a solution in terms of a modified Bessel function whose recurrence formulas produce all the results of special relativity. The Lorentz transformation maximizes the integral of the modified Bessel function and determines the paths of steepest descent in the classical limit. If the definitions of frequency and wave number in terms of the phase were used in special relativity, the condition that the frame be inertial would equate the superluminal phase velocity with the particle velocity in violation of causality. In order to get surfaces of constant phase to move at the group velocity, an integrating factor is required which determines how the intensity decays in time. The phase correlation between neighboring sites in quantum mechanics is given by the phase factor for the electron to reverse its direction, whereas, in special relativity, it is given by the Doppler shift.

Key words: Random Walks; Quantum Mechanics; Special Relativity; Ordinary and Modified Bessel Functions.

1. Paradoxes

Relativistic quantum mechanics originated by imposing the operator correspondence

$$\omega \leftrightarrow i\partial_t \quad \text{and} \quad \kappa \leftrightarrow -i\partial_q \qquad (1)$$

for the frequency ω and the wave number κ, on the relativistic dispersion relation

$$\omega^2 = (\kappa c)^2 + \varpi^2, \qquad (2)$$

where $\varpi = mc^2/\hbar$ is the rest mass frequency. This dispersion relation was derived from the expression for the relativistic energy by invoking the Planck, $E = \hbar\omega$, and de Broglie, $p = \hbar\kappa$ relations for the energy E and the momentum p. Introducing (1) into (2) results in

$$\Box\varphi := \left(\partial_q^2 - \frac{1}{c^2}\partial_t^2\right)\varphi = \left(\frac{\varpi}{c}\right)^2 \varphi, \qquad (3)$$

the Klein-Gordon equation, which was the starting point of Dirac's relativistic theory of the electron.

The union of special relativity and relativistic quantum mechanics has always been precarious. In special relativity there is only one velocity: the constant, relative velocity between two inertial frames. Wave mechanics identifies a variety of velocities, the two most important of which are the phase and group velocities. The former describes the propagation of individual waves in a wave packet while the latter takes into account the modulation due to the wave packet. The

product of the phase and group velocities is equal to c^2, so that if one is less than c the other will necessarily be greater than c. The particle velocity is associated with the group velocity, and the superluminal phase velocity is said to be devoid of any physical meaning [1, 2].

Soon after the development of special relativity, it was realized that the laws of dispersion in refractive media in a region of anomalous dispersion can give rise to phase and group velocities greater than c. Wien remarked that in the spectrum of a medium displaying anomalous dispersion, there can exist a region near the absorption line where the index of refraction is less than unity for which the velocity is greater than c. Subsequently, Laue [3] found that for natural light in a dispersive medium, characterized by the wavelength of its average intensity, the group velocity is the relevant quantity for the propagation of energy. Due to strong absorption, however, it is not possible to define a velocity of propagation over distances of several wavelengths so that the group velocity becomes ill-defined. Under these circumstances light can only be defined statistically, without a precise velocity of propagation. Sommerfeld [4] argued that the wave front must always propagate with a velocity not exceeding c. Hence, this velocity must be identified with the phase velocity, and not the group velocity.

The real problem lies with seeming incompatibility between the periodic and aperiodic nature of quantum phenomena and special relativity, respectively. In order to make the Lorentz transformation look like a rotation in space-time, one has to resort to the introduction of an imaginary angle together with an imaginary coordinate. The constant velocity of an inertial frame in special relativity has no counterpart in quantum mechanics. A particle velocity cannot even be defined: Schrödinger associated the group velocity of the wave packet with the *classical* particle velocity, where the particle trajectory is the classical limit of the wave packet. A dispersive wave train, however, has very little in common with deterministic particle trajectories. Dispersive wave trains tend to disperse and distort so that they will breakup as time goes on. They hardly resemble a particle trajectory.

The superluminal value of the phase velocity disappears when a continuum is replaced by a discrete, periodic lattice. This has also resolved the problem of non-locality, and the finiteness of signal transmission, thereby restoring causality by renouncing the possibility of knowing what goes on over distances smaller than the Compton wavelength. Due to the discreteness of the lattice masses there exists a smallest measurable wavelength; that is, wave properties can be determined at the lattice mass points and not in between them. Causality is ensured because of the threshold limit on measurability.

The transition between special relativity and non-relativistic quantum mechanics consists in replacing the real argument in the normalized modified Bessel function by an imaginary one, thereby obtaining an ordinary Bessel function together with space- and time-dependent phase factors that convert the recursive formulas into quantum mechanical recognizable expressions. Modified Bessel functions of the first kind are ubiquitous in time-dependent random walks, which are the discrete space analogs of continuous diffusion processes [5]. The space-dependent phase factor connecting the modified and ordinary Bessel functions is the Feynman prescription for weighting each turn of the electron in a checker board representation of the motion in one-space–one-time dimension. This phase factor converts the recurrence formula of the ordinary Bessel function into a finite difference differential (fdd) Schrödinger equation [6]. What was a probability density has now become a probability *amplitude*. The phase factors 'guide' the motion of the particle but do not contribute to the probability of finding particle at any given point in space-time. Therefore, Bessel functions, and the underlying stochastic processes of which they are solutions, provide a natural framework from which to compare relativistic quantum mechanics and special relativity.

The fdd Schrödinger equation contains relativistic corrections to nonrelativistic quantum mechanics [6]. Yet, the stability condition of the finite difference scheme [7] does not contain the velocity of light so that the formulation cannot encompass the ultrarelativistic limit. The stability condition of the finite difference scheme of the ultrarelativistic limit contains c, but the mass of the particle and Planck's constant have disappeared. Consequently, ultrarelativistic path trajectories will look quite different from their nonrelativistic counterparts. In the nonrelativistic limit, space and time scale like brownian motion, whereas they scale in the same way in the ultrarelativistic limit. Since the mass and Planck's constant have vanished in the stability criterion of the ultrarelativistic limit, they must be reintroduced by hand. Therefore, in the transition between nonrelativistic and ultrarelativistic quantum theory there is a change in the lattice spacing

so what is formally a parabolic equation will transform into a genuine hyperbolic one. The solutions to the resulting pair of first order fdd wave equations describe a right-handed (or magnetic moment parallel to the momentum direction) and a left-hand (or magnetic moment antiparallel to the momentum direction) particle of spin-$\frac{1}{2}$. This description is valid strictly in the zero-mass limit; otherwise, a Lorentz transformation could convert a left-handed massive particle into a right-handed one [8]. These equations are dispersion free; they become dispersive when the rest mass frequency is introduced through a time-dependent phase factor.

In this manner a pair of coupled first order equations is obtained that have the same dispersion characteristics as the second order Klein-Gordon equation. However, the phase factor is incompatible with a process whose solution is an ordinary Bessel function. We lose the picture of an electron propagating along a chain with a certain chirality, and reversing both its direction and handedness at random times. Yet, all that is required to make this picture reappear is a sign change in the mass term in the Klein-Gordon equation. It is precisely the aim of this paper to justify this sign change, thereby contrasting periodic from aperiodic propagation in a one-dimensional lattice. The former characterizes quantum phenomena while the latter pertains to special relativity. More specifically, the sign change of the mass term converts the Klein-Gordon equation into the Ehrenfest equation, whose solution is expressible in terms of a modified Bessel function, and whose recurrence relations give the relativistic expressions for energy, momentum, velocity, and the mass dependence upon the velocity. In the classical limit as $\hbar \rightarrow 0$, the Lorentz transformation determines the path of steepest descent in the integral representation of the modified Bessel function by maximizing its integrand. In this sense, the Lorentz transformation can be considered as optimal.

The constraint that the frame be inertial is completely foreign to the propagation of dispersive wave trains. It implies that the phase be constant resulting in a velocity of propagation equal to the phase velocity. But, because this velocity is superluminal, it is in contradiction with causality and the Lorentz transformation. New definitions of the frequency and wave number are called for in terms of the derivatives of the phase. They are distinguishable from the ordinary definitions only in the presence of a finite mass. The Pfaffian form of the phase will turn out to possess an integrating factor which is the solution of the characteristic equation for the wave intensity under the condition of constant velocity. The new definitions of frequency and wave number require new operator correspondences with the partial derivatives of time and space. The correct operator correspondence is entirely compatible with the Lorentz transformation if the Ehrenfest, and not the Klein-Gordon, equation is satisfied. Moreover, it provides the condition for the integrability of the phase so that it will propagate with the *group*, or particle, velocity, and not the phase velocity which is superluminal. In fact, the association of de Broglie waves with the relativistic motion of a particle is not warranted since there is a single, constant, velocity in special relativity. Finally, the phase correlation between any two neighboring lattice sites will be shown to obey the generalized Feynman prescription for a reversal of the spatial direction of an electron in the periodic case, while in the aperiodic case, it will be given by the radial Doppler effect. In the latter case, a subdivision of the original interval between the mass points leads to the relativistic law of the addition of velocities.

2. Discrete Relativistic Quantum Mechanics

A grid is constructed in the (q, t) plane such that the distance q is an integral number of times the fundamental interval, $\chi := \hbar/mc$, which is the Compton wavelength of a particle of mass m, divided by 2π. The time axis is divided into units ϖ^{-1}, which is the time it takes light to cross the particle's Compton wavelength. With the mesh ratio as $\mu := 1/\varpi\chi = 1/c$, the Courant-Friedrichs-Lewy stability condition [9] can be phrased as: in order that the closed interval, $[q - t/\mu, q]$, contain the point $\xi = q - vt$, the mesh ratio must satisfy $\mu v \leq 1$, or $\beta := v/c \leq 1$. The impossibility of a velocity greater than that of light, "whether it be the velocity of electronic or particle motion, or the propagation of an electrodynamic or mechanical signal" [10], will ensure that the domain of the difference equation contains the domain of dependence of the limiting differential equation.

Consider an 'honest' [5], continuous-time, one-dimensional random walk in which the time interval between successive jumps is regulated by a Poisson process, where the probability of a single jump is $e^{-\varpi t}$. The jumps themselves are random events which take on values of $+1$ and -1 with probabilities p and q (without requiring $p+q = 1$ because we will presently

turn them into probability amplitudes). Suppose that after n steps, the particle is found at $r \geq 0$ at time t. The only way that this can happen is that out of the n jumps, $\frac{1}{2}(n+r)$ had to have been positive, and $\frac{1}{2}(n-r)$ negative. This requires $n - r = 2k$ to be even. Thus, the probability to be in state $r \geq 0$ at time t is

$$
\binom{n}{\frac{1}{2}(n+r)} p^{\frac{1}{2}(n+r)} q^{\frac{1}{2}(n-r)} = \binom{r+2k}{r+k} p^{r+k} q^k .
$$

Since it is not certain that n jumps will occur in time t, this probability must be Poisson averaged over all possible numbers of jumps. Consequently, the probability to be at r at time t is

$$
p_r(t) = e^{-(p+q)\varpi t} \sum_{k=0}^{\infty} \frac{(\frac{1}{2}\varpi t)^{r+2k}}{(r+2k)!} \binom{r+2k}{r+k} p^{r+k} q^k
$$

$$
= \sqrt{(p/q)^r} \, e^{-(p+q)\varpi t} I_r(2\sqrt{pq}\varpi t), \qquad (4)
$$

where $I_r(\varpi t)$ is a modified Bessel function of the first kind. The generating function of the modified Bessel function

$$
\exp\left\{ \varpi t \left(pz + \frac{q}{z} \right) \right\} = \sum_{r=-\infty}^{\infty} I_r(2\sqrt{pq}\varpi t) \left(\sqrt{\frac{p}{q}} z \right)^r
$$

will give the correct normalization on setting $z = 1$ only when the *a priori* probability is conserved, *viz.*, $p + q = 1$.

The transition from discrete random walks in continuous-time to quantum mechanics converts the modified Bessel function of the first kind into an ordinary Bessel function of the first kind. Instead of the probability of a jump ϖt in time t, a probability amplitude $e^{i\pi/2}\varpi t$ is assigned for the reversal in the path in the same time interval [11]. For a particle that makes a transition from the origin to r in time t, this would produce a phase factor of $e^{ir\pi/2}$ since the particle can reverse its direction at any stage, independently of what has occurred previously. Evidently, this should lead to a type of motion that is formally similar to Brownian motion for equal *a priori* probabilities. This is what we would expect in the nonrelativistic limit [6]. However, when the particle travels at velocities comparable to c, it will show much less of a tendency to reverse direction at each lattice point. *A fortiori*, it will become evident from the recurrence relations of ordinary Bessel functions,

that the finite difference quotient does not use nearest neighbor spacings. Equal *a priori* probabilities can no longer be assumed for transitions to the left or right along the linear lattice, as was done in the derivation of the fdd Schrödinger equation [6] [*vid.* (33) ff].

The tendency not to reverse direction can be taken into account by applying the Feynman prescription to the *a priori* probability amplitudes. In attempting to geometrize the Dirac equation, Feynman used the prescription of loading each turn by a phase factor $e^{i\pi/2}$ [12]. Feynman's prescription, when applied to the *a priori* probability amplitudes, would give the expressions $p = \frac{1}{2}e^{-i\pi/2}$ and $q = \frac{1}{2}e^{i\pi/2}$ as the amplitudes for maintaining and reversing course, respectively.

These expressions for the *a priori* probability amplitudes and analytic continuation, $\varpi t \rightarrow e^{i\pi/2}\varpi t$, convert a modified Bessel function into an ordinary Bessel function, whose generating function is

$$
\exp\{-i\varpi t \sin\theta\} = \sum_{r=-\infty}^{\infty} I_r(i\varpi t) e^{-ir(\theta+\pi/2)}
$$
$$
= \sum_{r=-\infty}^{\infty} J_r(\varpi t) e^{-i\theta r} . \qquad (5)
$$

$J_r(\varpi t)$ is the ordinary Bessel function, $z = e^{-i\theta}$ and $\theta \neq n\pi$, where n is an integer. Since the wave amplitude is complex and $t > 0$, we can represent it by one of the complex conjugate Hankel functions, whose real part is the ordinary Bessel function, and also satisfies Bessel's equation and the recurrence formulas. The sum of the *a priori* probability amplitudes has cancelled the time-dependent phase factor, $e^{-i\varpi t}$, and the space-dependent phase factor $e^{ir\pi/2}$, which is responsible for the diffusive behavior of the motion, has been eliminated by the term $(p/q)^{r/2}$ in expression (4). These phase factors are responsible for converting the recursive formula of the ordinary Bessel function into a fdd equation whose long wavelength limit is the Schrödinger equation [6]. Roughly speaking, the paths get ironed out in the ultra-relativistic limit. Using the Feynman prescription, it is precisely the factor $e^{i\pi/2}$ that would be obtained each time the electron would reverse its direction. In a transition from the origin to the state r, the wave amplitude would acquire a cumulative phase factor $e^{ir\pi/2}$. Such a phase factor is hardly applicable to the motion of a particle whose speed is comparable to c because it will show very little tendency to reverse its direction. However, the same phase factor will reappear when

we transform from elastic to particle waves. The fact that $\sum_{r=-\infty}^{\infty} J_r^2(\varpi t) = 1$, guarantees that probability is conserved, independent of whatever phase factors may be present in the wave amplitude. External potentials, as well as the zero point of the relativistic energy, enter the quantum mechanical description through such time-dependent phase factors.

The ordinary Bessel function satisfies the second order recurrence relation

$$\psi_r''(\varpi t) = \frac{1}{4}\varpi^2 \left(\psi_{r-2}(\varpi t) - 2\psi_r(\varpi t) + \psi_{r+2}(\varpi t) \right),$$

(6)

where the prime means differentiating with respect to the argument. The presence of non-nearest neighbor sites requires the introduction of the symmetric difference quotient [13]

$$\hat{\partial}_q \psi_r = (2 \chi)^{-1}(\psi_{r+1} - \psi_{r-1}) = \frac{1}{2}\left(\partial_q + \bar{\partial}_q \right) \psi_r,$$

where ∂_q and $\bar{\partial}_q$ are the forward and backward difference quotients, respectively. Formula (6) can thus be written as the fdd wave equation

$$\left(\partial_t^2 - c^2 \hat{\partial}_q^2 \right) \psi_r = 0.$$

(7)

The wave amplitude satisfies either

$$\left(\partial_t + c \hat{\partial}_q \right) \psi_r = 0,$$

(8)

depending upon space and time only through the combination

$$\psi_r = \psi_R(q - ct),$$

or

$$\left(\partial_t - c \hat{\partial}_q \right) \psi_r = 0,$$

(9)

whose general solution is

$$\psi_r = \psi_L(q + ct).$$

The two uncoupled wave equations, (8) and (9), apply to the zero fermion mass limit[1]. It was suggested long ago that only one of the two states had to do with physical reality. The two component theory of the neutrino is one such example, where $\psi_L \neq 0$ and $\psi_R = 0$, or *vice versa* [8, p. 170].

If we want to take both directions of propagation into account simultaneously, without going to a second order wave equation, we can write the wave amplitude as a two component vector $\vec{\psi} = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$. The first order wave equations, (8) and (9), can then be written simultaneously as

$$\left(\partial_t + c\, \sigma_z \hat{\partial}_q \right) \vec{\psi} = 0,$$

(10)

where $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is a Pauli spin matrix. According to the usual interpretation, the subscripts refer to the zero mass limit, where ψ_R and ψ_L describe a right-handed, or spin parallel to the momentum direction, and a left-handed, or spin antiparallel to the momentum direction, of a spin-$\frac{1}{2}$ particle, respectively.

The wave amplitude can be represented as a complex cylindrical function

$$\psi_r(\varpi t) = \frac{1}{\pi} \int_C \exp\left\{ i\left[r\theta - \varpi t \sin\theta \right] \right\} \, d\theta, \quad (11)$$

where the contour C either begins between $\pi + i\infty$ and $\pi/2 + i\infty$ and ends between $-i\infty$ and $\pi/2 - i\infty$, or begins between $-\pi/2 - i\infty$ and $-i\infty$ and ends between $-\pi + i\infty$ and $-\pi/2 + i\infty$. Since we are interested in the classical limit, $\hbar \to 0$, we can use the method of steepest descent to obtain the leading term in the asymptotic expansion of (11) [14][2].

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[1] "It is amusing" [8] that Maxwell's equations in empty space have the same structure as the uncoupled wave equations, (8) and (9), with components $(B_i + e^{-i\pi/2}E_i)$, $i = 1, 2, 3$, where the difference quotients are multiplied by the three-dimensional spin matrices.

[2] For small, but finite, θ the exponent in (11) can be expanded in a power series. Retaining the lowest nonlinear term, a polynomial of degree three is obtained, giving the integral the form of an Airy integral

$$2\mathrm{Ai}\left(\sqrt[3]{2(\varpi t)^2}(\beta - 1) \right) = \int_{-\infty}^{\infty} \exp\left(i\left[\sqrt[3]{2(\varpi t)^2}(\beta - 1)z + \frac{1}{3}z^3 \right] \right) \, dz,$$

where $z = \sqrt[3]{\frac{1}{2}\varpi t}\,\theta$. The Airy function, Ai, drops rapidly to zero for $\beta > 1$, and is a rapidly oscillating, and slowly, decreasing function for $\beta < 1$. The diffraction pattern of 'dark' and 'light' regions,

The integrand of (11) can be written as the phase factor $\exp iS(\theta)$, where

$$S(\theta) = r\theta - \varpi t \sin\theta. \tag{12}$$

Because of causality, $\beta = r/\varpi t < 1$, (12) has a pair of saddle points located at

$$S'(\theta) = r - \varpi t \cos\theta = 0, \tag{13}$$

on the real axis. The positive saddle point, θ_+, takes on values from 0 to $\pi/2$, while the second saddle point, $\theta_- = -\theta_+$, varies between $-\pi/2$ to 0 [14]. If causality were violated, the saddle points would lie along the imaginary axis, and hence would not correspond to any observable phenomenon.

On account of mirror image symmetry, we need only consider the saddle point θ_+. Evaluating (12) at θ_+ results in

$$S(q,t) = (q/\chi)\cos^{-1}\beta - \varpi t\sqrt{1-\beta^2}, \tag{14}$$

which can be treated as a classical action in units of $\hbar$. Consequently, the wave number and frequency are defined by

$$\partial_q S = \chi^{-1}\cos^{-1}\beta = \kappa \tag{15}$$

and

$$-\partial_t S = \varpi\sqrt{1-\beta^2} = \omega, \tag{16}$$

respectively. In the limit $\beta \ll 1$, (15) and (16) do not reduce to the well-known dispersion relation of nonrelativistic mechanics. In particular, (15) does not provide a linear relationship between momentum and velocity in the $\beta \ll 1$ limit. This will be seen to be due to absence of the space-dependent phase factor $e^{ir\pi/2}$,

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corresponding to $\beta > 1$ and $\beta < 1$, respectively, is separated by a caustic at $\beta = 1$. For $\beta > 1$, there is no stationary point, while for $\beta < 1$ there are two. The curve of the index of refraction versus frequency presents a curious anomaly in an absorption band.

In terms of the index of refraction n,

$$\beta^{-1} = n + \omega\, d_\omega n,$$

so that β can become greater than one when the index of refraction becomes a decreasing function of the frequency. This happens in a region of absorption where the group and phase velocities can both be greater than c [15]. It would correspond to the dark region, where the group velocity would lose its significance as the signal velocity, or the velocity of energy transport. A large negative value of $d_\omega n$ is tantamount to a rapid variation of ω with κ so that the approximation used to derive the group velocity no longer holds.

whose presence is to shift the phase of the frequency and velocity by $\pi/2$, while the time-dependent phase factor, $e^{-i\varpi t}$, will contribute a zero-point term to the frequency [vid. (38) and (39)].

Expression (16) is equal to the trace of the energy-momentum tensor. The expression for the total energy of the system, $\hbar\omega$, can be considered as the relativistic generalization of the virial theorem [16]. In the non-relativistic limit, (16) becomes $\hbar(\omega - \varpi) \approx -mv^2/2$: the total energy (less the rest energy) is equal to the negative kinetic energy. The corresponding virial theorem is $2T = \eta U$, where T is the kinetic energy, and U is potential energy, which is a homogeneous function of degree $\eta = -1$, implying an inverse law, like the Coulomb potential. Consequently, (16) is the relativistic generalization of the virial theorem, and implicit in it is the fact that the particles are interacting through a potential which varies inversely with the distance of their separation. The relativistic virial theorem also applies to the mass defect and the red shift in the presence of the gravitational potential, which has the same inverse dependency on the radius as Coulomb's law.

The consistency condition,

$$\partial_t \kappa + \partial_q \omega = 0 \tag{17}$$

is not only the exactness condition of the state function (14), but it is moreover a statement of wave conservation [17]. Expressions (15) and (16) give the group velocity as

$$\partial_\kappa \omega = c\cos\theta = v, \tag{18}$$

since

$$\omega = \varpi \sin\theta. \tag{19}$$

These relations show little in common with particle properties because the frequency is an odd function of the wave number, the velocity does not change sign with the wave number, and large velocities correspond to small wave numbers. Rather, (19) is reminiscent of the dispersion equation for elastic waves in a one-dimensional crystal whose frequency spectrum has an upper cutoff frequency $\omega_{\max} = \varpi$, corresponding to a minimum wavelength 4χ.

Both the wave number and frequency propagate at the group velocity, v, so that the frequency equation can be written as

$$\partial_t \left(\omega a^2\right) + \partial_q \left(v\omega a^2\right) - \omega\left(\partial_t a^2 + v\partial_q a^2\right) = \frac{\omega}{t}a^2, \tag{20}$$

where $a(q, t)$ is the amplitude of a slowly varying wave train in space and time. The ratio of the energy flux, $v\omega a^2$, to the energy density, ωa^2, is group velocity, v, which can be taken as its definition. Energy conservation demands [17]

$$\partial_t \left(\omega a^2\right) + \partial_q \left(v\omega a^2\right) = 0, \qquad (21)$$

so that (20) is reduced to the characteristic form of the amplitude equation

$$d_t\, a^2 = -\frac{a^2}{t}, \qquad (22)$$

on $q/t = v$. Integrating (22) along the characteristic yields

$$a = \frac{\mathcal{A}}{\sqrt{t}}, \qquad (23)$$

where the complex constant of integration, $\mathcal{A}$, must be determined by some other method. The time-dependency of the intensity $|a|^2$, that is attributed to the non-uniformity of the wave train, is precisely the quantum mechanical probability for a free-particle.

The method of steepest descent can be used to determine the complex constant in (23) in the classical limit as $\hbar \to 0$. Since $S'' = -\varpi t \sin \theta \neq 0$ the saddle points are simple and the procedure is straightforward [14]. The leading term in the asymptotic expansion is

$$\psi_r(\varpi t) \sim \frac{\exp\left\{-i\left(\omega t - r \cos^{-1}\beta - \tfrac{1}{4}\pi\right)\right\}}{\sqrt{\tfrac{1}{2}\pi\omega t}}, \qquad (24)$$

which identifies the complex constant of integration in (23) as $\mathcal{A} = e^{i\pi/4}/\sqrt{\omega/2}$.

No superluminal phase velocities are found here as they appear in special relativity. The dispersion equation is similar to that of light waves, $\omega = c\kappa/n(\kappa)$, where the index of refraction $n(\kappa) = 1/\mathrm{sinc}\,\theta$. The function $\mathrm{sinc}\,\theta := \sin\theta/\theta$ is a low-pass filtering function whose spectrum is flat up to a cutoff value $\kappa_{\max} = \pi/2\chi$, thereby eliminating wavelengths smaller than 4χ. The corresponding cutoff frequency is $\omega_{\max} =$

ϖ^3. Since $\mathrm{sinc}\,\theta$ does not exceed unity, the index of refraction $n(\kappa)$ is greater than unity, thereby ensuring causality.

In this case the phase velocity, $V = c/n(\kappa)$, will be less than c. This corresponds to the propagation of elastic waves in a one-dimensional lattice [20, 21], and is at odds with the relativistic result, $V \cdot v = c^2$. Here, the product of the phase and group velocities $V \cdot v = c^2\mathrm{sinc}\,2\theta \leq c^2$. The former leads to the inescapable conclusion that if the group velocity is less than c, the phase velocity must be greater than c [1]. The latter asserts that both the phase and group velocities approach the limiting velocity c in the limit of infinite wavelength as $\theta \to 0$. In this limit, the lattice may be regarded as a continuum so that the velocity of propagation is independent of the wavelength [20]. At all finite wavelengths, both the phase and group velocities are less than c. In particular, at the edge of the first Brillouin zone, $\theta = \pi/2$, the phase velocity is $2c/\pi$, while the group velocity vanishes. As in the theory of band structures, no physical significance is attached to the phase velocity [22] so that the electron actually slows down as it approaches the edge of the Brillouin zone. This is reasonable since the disturbance represents a standing wave at the band edge. At the Brillouin zone boundary, the particle gets Bragg reflected and appears at the opposite edge with a negative wave number. However, the group velocity is still positive because (18) is an even function. For particle

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[3]This limit can be understood in terms of the Lamb shift where the equation of motion of a free electron in a fluctuation field would be infinite were it not for the spatial dependence of a given Fourier component of the electromagnetic field. Not only would the electron respond to a transverse oscillation of the electromagnetic field, but also to a longitudinal recoil. This recoil would be important for light with wavelengths smaller than the Compton wavelength of the electron, thus justifying the upper cutoff at the Compton wavelength [18]. Moreover, there will also be a lower cutoff because the electron will also experience very large, but very slow, low frequency fluctuations. The fluctuations will be suppressed by some type of binding giving rise to the lower limit, $\kappa_{\min}$. Bethe [19] assumed that $\kappa_{\min}$ could be determined from the average excitation energy of the $2s$ electron in the hydrogen atom.

The expression for the phase velocity was first derived by Baden-Powell in 1841 [20]. However, it was Lord Kelvin who in 1881 first realized that not only the phase velocity would be a function of the wave number, but, so too, would be the frequency. In so doing, he discovered the first low-pass filter, and developed an analogy between the propagation of elastic waves along a string and the propagation of electromagnetic radiation. The first mechanical filter was built by Vincent in 1898, which manifested anomalous dispersion for large damping. The history of wave propagation along a one-dimensional lattice of point masses can be found in Chapter 1 of [20].

behavior, negative values of the group velocity should be aligned with negative values of the wave number. This will be remedied in the next section where we will consider particle wave propagation that is out of phase with the propagation of elastic waves.

3. Relativistic Mass

The phase transformation $\vec{\psi} = \vec{\varphi}e^{i\sigma_x\varpi t}$, where $\vec{\varphi} = \begin{pmatrix} \varphi_R \\ \varphi_L \end{pmatrix}$, converts the wave equation (10) into

$$\left(\partial_t + c\sigma_z\hat{\partial}_q + i\varpi\sigma_x\right)\vec{\varphi} = 0, \tag{25}$$

where $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is another Pauli spin matrix. The wave equation (25) is the one-dimensional Dirac equation in the Weyl representation [8, p. 84]. The two components of the wave amplitude are the chirality components of the positive energy solution. Eliminating one of the chirality components in the Dirac equation (25) gives back the Klein-Gordon equation (3). It is commonly believed that (3) is (2) under the operator correspondence (1). Therefore, (3) should be completely compatible with special relativity.

However, the Klein-Gordon equation (3) does not admit a solution in terms of Bessel functions. If instead the transformation $\vec{\psi} = \vec{\varphi}e^{-\sigma_x\varpi t}$ is applied to the wave equation (10) there results

$$\left(\partial_t + c\sigma_z\hat{\partial}_q - \varpi\sigma_x\right)\vec{\varphi} = 0,$$

corresponding to the second order fdd equation

$$\left(\partial_t^2 - c^2\hat{\partial}_q^2\right)\varphi = \varpi^2\varphi \tag{26}$$

for either component. Equation (26) differs from the Klein-Gordon equation (3) by a change in sign of the mass term. This equation was originally analyzed by Ehrenfest [23], in an attempt to distinguish among the different wave velocities. Physically, (26) describes a string that is displaced from its equilibrium position by a force proportional to $\varpi^2\varphi$. The string is unstable since a disturbance propagates at the 'front' velocity c, and whose amplitude progressively increases in time [15]. By definition, the front is the surface beyond which, at any given instant in time, the medium is completely at rest [15, p. 11]. On the strength of the operator correspondence (1), the dispersion relation

$$\omega^2 = (\kappa c)^2 - \varpi^2 \tag{27}$$

shows that only wavelengths $< \chi$ would exhibit a real phase velocity less than c. But, such a possibility is excluded by the fact that the best localized state that can be constructed from a wave packet cannot contain plane-wave components with wavelengths smaller than the Compton wavelength of the particle. That the product of the phase and group velocities equals c^2 makes the group velocity greater than c, which is the velocity at which the front of the disturbance propagates. Then how is it possible for the group velocity to be greater than the velocity of the front of the disturbance?

The signal velocity, or the velocity at which the main signal arrives, is usually associated with the group velocity. However, in a region of anomalous dispersion either the group, or phase, velocity can become greater than c. In this region, the group velocity loses its meaning as a signal velocity, while the phase velocity becomes equal to that of light [4]. Sommerfeld concluded that in regions of anomalous dispersion, there does not exist a precise velocity of light, which needs to be defined *statistically*. The desire to associate de Broglie waves with particle propagation leads to the introduction of more than one velocity. However, special relativity identifies a single velocity, the particle velocity, or the constant, relative velocity with which one inertial frame moves with respect to another.

The Ehrenfest equation (26) does not admit periodic solutions over the admissible region of wavelengths $> \chi$. In this region it can, however, be solved in terms of modified Bessel functions, representing growing and decaying excitations in time. Modified Bessel functions of the second kind are infinite at $t = 0$, while those of the first kind are infinite at $t = \infty$. Ehrenfest's equation can be, and was probably, derived from the fdd telegrapher's equation [24]

$$\left(\partial_t^2 + 2\varpi\partial_t - c^2\hat{\partial}_q^2\right)\psi = 0 \tag{28}$$

through the same transformation, $\psi = \varphi e^{-\varpi t}$ [25]. The physical process described by the fdd telegrapher's equation is that of a particle propagating at velocity c, which at random times flips its direction of propagation and handedness at a constant rate, ϖ [26]. The probability for the particle to reverse its direction in time t is ϖt, while the probability that it will maintain its direction of motion is $(1 - \varpi t)$. If $p_R(q, t)$ is the probability for the particle to be at q at time t and moving to the right, and $p_L(q, t)$ is

the analogous expression for a particle moving to the left, then the transformations $p_i(q, t) = e^{-\varpi t}\tilde{p}_i(q, t)$ for $i = R, L$, lead to the coupled first order equations for the transformed probabilities

$$\begin{pmatrix} \partial_t + c\hat{\partial}_q & -\varpi \\ -\varpi & \partial_t - c\hat{\partial}_q \end{pmatrix} \begin{pmatrix} \tilde{p}_R \\ \tilde{p}_L \end{pmatrix} = \vec{0}.$$

Elimination of either probability results in the second order Ehrenfest equation (26). Consequently, the Ehrenfest equation has more to do with probabilities than with probability amplitudes. Only when the mass is allowed to become imaginary, $\varpi \to i\varpi$, does the Ehrenfest equation transform into the Klein-Gordon equation. Although the conversion of a real mass into an imaginary one is claimed to be warranted by analytic continuation [25], it loses contact with a physical, random walk process, and this is the why the Klein-Gordon equation does not admit solutions in terms of ordinary Bessel functions.

Since the symmetric finite-difference quotient $\hat{\partial}_q$ has the Fourier component $-i \chi^{-1} \sin\theta$ [13], the Ehrenfest dispersion equation (27) can be written as

$$-\omega^2 = \varpi^2 \cos^2\theta. \tag{29}$$

As opposed to the dispersion equations for the wave equation

$$\omega^2 = \varpi^2 \sin^2\theta \tag{30}$$

and the Klein-Gordon equation

$$\omega^2 = \varpi^2 \left(1 + \sin^2\theta\right),$$

the frequencies in the dispersion relation (29) cannot be real.

It is interesting to compare the dispersion relation (29) with the case where the *a priori* probability is conserved [6]. In that case, the space-dependent phase factor $e^{i\pi r/2}$ converts the generating function of the ordinary Bessel function (5) into

$$\exp\{i\varpi t \cos\theta\} = \sum_{r=-\infty}^{\infty} J_r(\varpi t)e^{-ir(\theta-\pi/2)}. \tag{31}$$

The time-dependent phase factor $e^{-i\varpi t}$ then transforms the generating function (31) into the characteristic function of a compound Poisson process [27]

$$\sum_{r=-\infty}^{\infty} e^{i\theta r}\psi_r(\varpi t) = \exp\{i\varpi t(\cos\theta - 1)\}. \tag{32}$$

The characteristic function of random jumps ± 1 with equal *a priori* probabilities is $\cos\theta$. Since the jumps are independent, the characteristic function of n jumps is $\cos^n\theta$. And because the jumps are Poisson regulated, this characteristic function must be averaged over all possible numbers of jumps that can occur up to time t, thus resulting in (32).

When the number of jumps is reduced by a factor of $1/n$ while their intensity is increased by n^2 and $n \to \infty$, the generating function (32) transforms into $e^{-\frac{1}{2}i\theta^2\varpi t}$, which is the central limit theorem result, or Wiener-Lévy process, [6]. It is also the quantum mechanical momentum probability amplitude in the nonrelativistic limit.

Differentiating (32) with respect to t and equating the coefficients of $e^{ir\theta}$ on both sides of the identity results in the Schrödinger fdd equation [6]

$$\partial_t \psi_r(\varpi t) = \frac{1}{2}i\varpi \left(\psi_{r-1}(\varpi t) - 2\psi_r(\varpi t) + \psi_{r+1}(\varpi t)\right)$$

$$= \frac{i\hbar}{2m}\partial_q \bar{\partial}_q \psi_r(\varpi t), \tag{33}$$

where $\psi_r(\varpi t) = J_r(\varpi t)e^{i(\pi r/2 - \varpi t)}$. The spatial phase factor $e^{ir\pi/2}$ introduces i into the fdd Schrödinger equation, while the temporal phase factor $e^{-i\varpi t}$ transforms the right-hand side of the usual recurrence formula

$$J_r'(\varpi t) = \frac{1}{2}\varpi \left(J_{r-1}(\varpi t) - J_{r+1}(\varpi t)\right)$$

into a second order difference quotient. Without these phase factors, a reiteration of this recurrence formula would give the second order recurrence formula (6) for elastic waves.

The wave amplitude is essentially the inverse transform of (31),

$$\psi_r(\varpi t) = e^{-i\varpi t}\frac{1}{\pi}\int_C \exp\left\{i\left[\varpi t \cos\theta + r\left(\theta - \frac{\pi}{2}\right)\right]\right\} d\theta, \tag{34}$$

which is one of two types of complex conjugate Hankel functions that are distinguished by the choice of the contour C. The contour C either begins between $\pi/2 - i\infty$ and $\pi - i\infty$ and ends between $\pi + i\infty$ and $3\pi/2 + i\infty$, or it begins between $-\pi/2 + i\infty$ and $i\infty$

and ends between $-\infty$ and $\pi/2 - i\infty$. Again, as the result of causality, the saddle points of

$$S(\theta) = \varpi t \left(\cos\theta - 1\right) + r\left(\theta - \frac{\pi}{2}\right) \qquad (35)$$

will lie on the real axis. The saddle points θ_+ and θ_-, defined by

$$S'(\theta) = -\varpi t \sin\theta + r = 0, \qquad (36)$$

are both positive, unlike the previous case of elastic waves propagating in a one-dimensional lattice. The simple saddle point, θ_+, takes on values between 0 and $\pi/2$, while θ_- varies between $\pi/2$ and π [4]. Evaluating (35) at the saddle points, $\theta_\pm$, gives the action

$$S(\theta_\pm) = \pm\left(\varpi t\sqrt{1-\beta^2} - (q/\chi)\cos^{-1}\beta\right) - \varpi t. \quad (37)$$

The frequency

$$\begin{aligned}-\partial_t S = \omega &= \varpi\left(1 \mp \sqrt{1-\beta^2}\right)\\ &= \varpi\left(1 - \cos\theta\right) = 2\varpi\sin^2\frac{1}{2}\theta\end{aligned} \qquad (38)$$

is an even function of the wave number, and the frequency difference, $(\omega - \varpi)$, is out of phase with the frequency of the elastic waves (19). The negative sign corresponds to $0 < \theta < \pi/2$, while the positive sign refers to the interval $\pi/2 < \theta < \pi$. The conservation of the *a priori* probability has retained two phase factors: the space-dependent phase factor, $e^{ir\pi/2}$, which shifts the phase θ by $\pi/2$, and the time-dependent phase factor, $e^{-i\varpi t}$, which adds a constant term to the frequency (38).

The group velocity

$$\partial_\kappa\omega = v = c\sin\theta \qquad (39)$$

coincides with the condition (36) that determines the saddle points, $\theta_\pm$, and is an odd function of κ. Negative wave numbers necessarily imply negative group velocities [22, p. 120]. The particle velocity vanishes in the limit as $\kappa \to 0$, in full accordance with the de Broglie relation for matter waves. The de Broglie

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[4]The additional saddle points located at $\theta = \theta_\pm + 2n\pi$, where n is an integer are inadmissible because of the impossibility of replacing the original contours $\mathcal{C}$, which lie in the strip $-\pi/2 < \theta < 3\pi/2$, by a combination of steepest descent contours emanating at any other saddle point other than $\theta_\pm$ [28].

relation itself, $mv = \hbar\kappa$, is now seen as a first order approximation to (39) in the limit $\theta \ll 1$, corresponding to the long wavelength limit. Whereas the group velocity of elastic waves has its maximum at long wavelengths, the particle velocity maximum occurs at 4 χ. The minimum wavelength that the string can support is 2 χ corresponding to the cutoff frequency $\omega_{\max} = 2\varpi$ of pair creation.

If the saddle point θ_+ is associated with a particle, the saddle point, θ_-, represents its antiparticle, or 'hole'. According to (38), antiparticle energies lie above particle energies. Particles and antiparticles travel in opposite directions. This can be seen by evaluating the complex integral (34) in the limit as $\hbar \to 0$ by the method of steepest descent. Since $S''(\theta_\pm) = \mp\sqrt{1-\beta^2} \neq 0$, the saddle points are simple, where the angles of inclination of the paths at these points are $\phi(\theta_+) = -\pi/4$ and $\phi(\theta_-) = \pi/4$. The expressions for the wave amplitude (34)

$$\psi_r^{(\pm)}(\varpi t) \sim \sqrt{\frac{2}{\pi\varpi t\sqrt{1-\beta^2}}} \qquad (40)$$

$$\cdot \exp\left\{\pm i\left[\varpi t\sqrt{1-\beta^2} - r\left(\frac{1}{2}\pi - \kappa\,\chi\right) - \frac{1}{4}\pi\right] - i\varpi t\right\}$$

represent travelling waves in the forward and backward directions in two non-overlapping frequency intervals. These are, in fact, the leading terms in the asymptotic expansion of the integral (34) as both r and $\varpi t \to \infty$, such that their ratio $\beta < 1$ [14]. The behavior of (40) is analogous to propagating particles forward in time and antiparticles backward in time [8, pp. 237–239]. For $\beta > 1$, the saddle points lie on the imaginary axis [28, p. 274], and therefore do not represent any observable phenomenon.

As in the band theory of metals, the effective mass can be defined as [22, p. 127]

$$m^\star := -\hbar\left(\partial_\kappa^2\omega\right)^{-1} = m\sec\theta = \pm\frac{m}{\sqrt{1-\beta^2}}, \quad (41)$$

where $\pm$ refers to the intervals $0 < \theta < \pi/2$, and $\pi/2 < \theta < \pi$, respectively. Increasing the particle velocity from zero, increases the mass until the limiting value of c is reached, corresponding to the wave number $\pi/2\,\chi$, or a wavelength 4χ. For larger values of the wave number, the effective mass (41) becomes *negative*. The velocity actually decreases with a negative effective mass until the particle reaches the top of

the band at wave number π/χ. Since this corresponds to a minimum wavelength 2χ, the Brillouin zone is twice as large as in the previous case of elastic waves. Nearest neighbor spacing allows the string to support a minimum wavelength which is half the size of the minimum elastic wavelength because the latter uses next to nearest spacings to define the (symmetric) difference quotient. At the top of the band, the particle gets Bragg reflected, appearing at the opposite edge of the first zone with a negative wave number, but with the same positive cutoff frequency, $\omega_{max} = 2\varpi$. The wave number will now increase until the particle is finally brought to rest at $\kappa = 0$, where the entire process is ready to be repeated. Thus, the particle will appear to oscillate backwards and forwards about a mean position, having its motion reversed when the wave number changes sign because its velocity will also change sign according to (39). Thus, it will spend half its time as a particle with positive effective mass, and the other half of its time as an antiparticle with a negative effective mass[5]. This appears very similar to the phenomenon of *Zitterbewegung* which is commonly attributed to the interference of positive- and negative-energy plane wave components of a bound state wave amplitude [8, p. 139].

4. From Relativistic Quantum Mechanics to Special Relativity

If the frequency in (29) is imaginary so, too, will the angle θ. Equating θ with an 'imaginary angle of rotation', just as is done in special relativity, (i. e., $\theta = i\vartheta$ and, consequently, $\cos(i\vartheta) = \cosh\vartheta$ and $\sin(i\vartheta) = i\sinh\vartheta$, with ϑ real), converts a true rotation into a Lorentz transformation. The fact that the last term in the Ehrenfest equation, (26), has transformed the recurrence formula for the ordinary Bessel function (6) into the recurrence relation

$$I_r''(\varpi t) = \frac{1}{4}\varpi^2 \left(I_{r+2} + 2I_r + I_{r-2} \right), \tag{42}$$

for the modified Bessel function, adds further support to treating θ as an imaginary angle [cf. (63) and (69)]. Both the modified Bessel function of the first kind,

[5]As far back as 1930, Heisenberg [29] postulated that, in the neighborhoods of the minima and maxima of the standing wave solution to the fdd Klein-Gordon equation, the electron would behave as a normal electron and proton, respectively. Heisenberg's proton was actually Dirac's positron.

I_r, and the second kind, K_r, satisfy the recurrence formula, (42). Other recurrence formulas are related through time reversal, $t \rightarrow -t$. The time reversal symmetry of the modified Bessel functions will be discussed below [cf. (60) and (59), respectively].

Consider two inertial frames O and O', the latter travelling at a constant velocity v with respect to the former. The particle is placed at the origin of the O' system. The motion, considered in the O frame, is related to O' by the Lorentz transformation

$$q = ct' \sinh\vartheta, \; t = t' \cosh\vartheta. \tag{43}$$

The angle ϑ can only depend on the relative velocity. Dividing the former by the latter results in

$$\beta := \frac{q}{ct} = \tanh\vartheta, \tag{44}$$

where q/t, at the origin of O', is the velocity v of the frame O' relative to O. The logarithmic expression for the inverse hyperbolic tangent can be used to express the exponent of the angle as

$$e^\vartheta = \sqrt{\frac{1+\beta}{1-\beta}}, \tag{45}$$

which is related to the radial Doppler effect [30]. This is the only type of Doppler effect that can occur in a single dimension. The electron, or the radiation source, is at rest in the frame O'. The frequency of radiation emitted in the frame O' will be different from the frequency measured in the frame O by precisely the factor (45) [vid. (93) ff].

The relationship between the Lorentz transformation (43) and the modified Bessel function of the second kind can be discerned by writing the latter in the approximate form

$$K_r(\varpi t') = \int_0^\infty \exp\left(-\varpi t' \cosh\vartheta\right) \cosh r\vartheta \, d\vartheta$$
$$\approx \frac{1}{2}\int_0^\infty e^{r\vartheta - \varpi t' \cosh\vartheta} \, d\vartheta, \tag{46}$$

since the term $e^{-r\vartheta}$ decreases with increasing ϑ, and is negligible in comparison with $e^{r\vartheta}$ when r is large [31]. The asymptotic approximation to the modified Bessel function of the second kind, (46), can be obtained from the method of steepest descent.

The function

$$S(\vartheta) = r\vartheta - \varpi t = r\vartheta - \omega t' \tag{47}$$

determines that the path of steepest descent lies along the real axis

$$S'(\theta) = r - \varpi t' \sinh \vartheta = 0. \tag{48}$$

This path is none other than the first equation in the Lorentz transformation, (43). The stationary condition and $S''(\vartheta) = -\varpi t' \cosh \vartheta < 0$ render the integrand a maximum, and it can be thought of as determining the optimal value of the 'imaginary angle of rotation' as given by the Lorentz transformation. *The Lorentz transformation is the condition for the maximum of the integrand of the modified Bessel function in the classical limit as $\hbar \to 0$ [30]. Whereas the ratio q/t is the velocity, the shifted ratio q/t' determines mass dependence on the velocity [30].*

Substituting (43) into (47) gives the action

$$S(r, t') = r \sinh^{-1}\left(\frac{q}{ct'}\right) - \varpi t'\sqrt{1 + (q/ct')^2} \tag{49}$$

in units of $\hbar$. The total energy (in units of Planck's constant), defined in terms of the action (49),

$$-\partial_{t'} S = \varpi \sqrt{1 + (q/ct')^2} = \omega, \tag{50}$$

has to be the same as the total relativistic energy

$$E/\hbar = \omega = \varpi \sqrt{1 + (p/mc)^2} = \varpi \cosh \vartheta. \tag{51}$$

This identifies the momentum as

$$p = m\frac{q}{t'} = mc \sinh \vartheta, \tag{52}$$

or, since $p = \hbar \kappa$, the wave number is

$$\kappa = \chi^{-1} \sinh \vartheta. \tag{53}$$

In quantum mechanics, the angle θ is proportional to the wave number, and the relative velocity is related to it by a circular function [*vid.* (39)]; on the contrary, in special relativity, the angle ϑ is a function of the relative velocity, and determines the wave number through a hyperbolic relation [*vid.* (53)].

Clocks in the frame O' run slower than those in O, $t > t'$, whereas the characteristic frequencies in O' are larger than in O, $\omega > \varpi$. These characteristic frequencies, or energies, are not the clock frequencies, which are diminished by the motion on account of the Doppler effect, but those of the wave associated with this motion [32]. The products of the characteristic

frequencies and times in the frames O and O' are invariant, *i.e.*, $\omega t' = \varpi t$. This relation can be used to eliminate t' in favor of t in (52) with the result that

$$p\frac{c^2}{E} = \frac{q}{t} = v.$$

Substituting this into (51) yields the relativistic mass dependency on the velocity [*cf.* the effective mass (41) in the periodic case]

$$m(v) = m \cosh \vartheta = \frac{m}{\sqrt{1 - \beta^2}}. \tag{54}$$

The 'imaginary angle of rotation' is defined by

$$\partial_r S = \sinh^{-1}(\kappa \chi) = \vartheta. \tag{55}$$

The consistency condition (17) is easily verified by differentiating (50) with respect to r,

$$\partial_r \partial_{t'} S = -\chi \partial_q \omega = -c \sinh \vartheta \, \partial_q \vartheta, \tag{56}$$

and (55) with respect to t',

$$\partial_{t'} \partial_r S = \partial_{t'} \vartheta = \cosh \vartheta \, \partial_t \vartheta = \chi \partial_t \kappa, \tag{57}$$

where the second of the Lorentz transform relations (43) has been used. Equating these cross second derivatives immediately yields (17). Substituting the expressions for the frequency, (50), and wave number, (53), into the consistency condition, (17), gives

$$-\frac{\partial_t \vartheta}{\partial_q \vartheta} = \partial_\kappa \omega = c \tanh \vartheta \tag{58}$$

as the 'group' velocity, although no wave packet has been contemplated. According to (44), (58) is the relative velocity of the frame O' with respect to O. Yet, if one were to apply the usual definitions of frequency, $-\partial_t \vartheta$, and wave number, $\hat{\partial}_q \vartheta$, one would be forced to conclude that (58) is the phase velocity, ω/κ, which cannot be equal to the group velocity, $\partial_\kappa \omega$, in a dispersive medium like the one under consideration [*vid.* (83) ff].

A straightforward application of the method of steepest descent gives the leading term in the asymptotic expansion of the modified Bessel function of the second kind, (46), as

$$K_r(\varpi t') \sim \sqrt{\frac{\pi}{2\varpi t'}} \frac{e^{r \sinh^{-1}(q/ct') - \omega t'}}{\sqrt[4]{1 + (q/ct')^2}}. \tag{59}$$

In contradistinction, the modified Bessel function of the first kind is asymptotically given by

$$I_r(\varpi t') \sim \frac{1}{\sqrt{2\pi \varpi t'}} \frac{e^{-r\sinh^{-1}(q/ct')+\omega t'}}{\sqrt[4]{1+(q/ct')^2}}. \quad (60)$$

The asymptotic expressions for the modified Bessel functions (59) and (60) contain aperiodic excitations, which exponentially decay and grow in time, respectively. Expression (60) can be used to discriminate between analytic continuation, $\varpi t \rightarrow e^{i\pi/2}\varpi t$, that causes $I_r(i\varpi t) = e^{ir\pi/2}J_r(\varpi t)$, and the conversion of the mass into an imaginary quantity, $\varpi \rightarrow i\varpi$, which transforms the Ehrenfest equation (26) into the Klein-Gordon equation (3).

If we begin with the telegrapher's equation (28) and introduce an imaginary time variable, then the phase transformation $\psi = \varphi e^{-i\varpi t}$ will transform it into the Klein-Gordon equation if c, which is a velocity, picks up a factor i as well [25]. Alternatively we can introduce an imaginary mass and the same phase transformation will yield the Klein-Gordon equation without analytic continuation. Now, if analytic continuation is applied to (60), and the distinction is dropped between t' and t, there results

$$J_r(\varpi t) \sim \frac{1}{\sqrt{2\pi \varpi t}} \frac{e^{i\left(\varpi\sqrt{1-\beta^2}\,t - r\cos^{-1}\beta - \frac{1}{4}\pi\right)}}{\sqrt[4]{1-\beta^2}},$$

which is the asymptotic expression for the Hankel function of the first type [28, p. 271 formula (7.2.35)]. If instead, we make the mass imaginary, we have to know that the coordinate q is related to the mass by $q = r\,\lambda$, which makes no sense as far as the original fdd telegrapher's equation (28) is concerned. This is the reason why a solution of the Klein-Gordon equation (3) does not exist in terms of an ordinary Bessel function.

The exponentially decaying and growing solutions display a symmetry in past and future. However, only (60) represents a probability density in the diffusion limit, $q \ll ct'$. In this limit (60) reduces to the Wiener-Lévy 'density'

$$I_r(\varpi t')e^{-\varpi t'}\,\chi^{-1} \sim \frac{e^{-q^2/4Dt'}}{\sqrt{4\pi Dt'}}, \quad (61)$$

where $D = \hbar/2m$ is the diffusion coefficient [vid. (33)], and

$$e^{-\varpi t'}\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\varpi t'\right)^{r+2k}}{(r+2k)!}\binom{r+2k}{r+k}$$

$$= e^{-\varpi t'}I_r\left(\varpi t'\right) = p_r\left(t'\right)$$

is the probability density of Brownian motion [5, p. 99]. The formation of the probability density (61) can be viewed as the result of the hyperbolically increasing Bessel function $I_r(\varpi t)$ being overpowered by an even more decreasing exponential factor $e^{-\varpi t'}$ to produce a sharply peaked probability density.

If we do not make a distinction between the two times and invoke analytic continuation, (61) becomes

$$\psi_r(\varpi t) = J_r(\varpi t)e^{i(r\pi/2 - \varpi t)} \sim \frac{e^{imq^2/2\hbar t}}{\sqrt{2\pi i\hbar t/m}},$$

which is the nonrelativistic free-particle kernel in the Feynman path integral formulation. The corresponding asymptotic expression for the modified Bessel function of the second kind,

$$K_r(\varpi t')e^{\varpi t'}\,\chi^{-1} \sim \sqrt{\frac{\pi}{4Dt'}}e^{q^2/4Dt'}, \quad (62)$$

can be considered as a diffusion process proceeding backward in time. The analytic continuation of (62) can be considered as the nonrelativistic free-particle propagator for an anti-particle, or a particle travelling backward in time [33]. The exponential decaying factor in time in (61) will be shown to determine the decay of the intensity, and to be the integrating factor for the phase when the constraint that the frame be inertial is imposed [vid. (89)]. The exponential growing solution will likewise be related to a phase reversal.

5. Quantum Mechanics versus Special Relativity

The distinction between relativistic quantum mechanics and special relativity can be scrutinized still further by considering the propagation of a disturbance along a string. The fdd equation, corresponding to the wave equation (10), is

$$\ddot{q}_r = \frac{1}{4}\varpi^2\left(q_{r+2} + q_{r-2} - 2q_r\right), \quad (63)$$

which is also the same as the recurrence relation of the ordinary Bessel function, (6). Equation (63) can be solved formally by setting

$$q_r = e^{-\vartheta r} q_0 \tag{64}$$

to give the subsidiary equation

$$\wp^2 = \frac{1}{2}\varpi^2 \left(\cosh 2\vartheta - 1\right) = \varpi^2 \sinh^2 \vartheta. \tag{65}$$

Once the $\wp$ operator has been written for the time derivative, it can be subsequently treated as if it were a number [31]. There are two equal and opposite *real* values of ϑ for *real* $\wp$.

The $\wp$-multiplied Laplace transform of q_r is [34]

$$\wp \int_0^\infty e^{-\wp t} q_r \, \mathrm{d}t = \exp\left(-r \sinh^{-1}(\wp/\varpi)\right) q_0.$$

With the aid of the corresponding inverse Mellin transform we get

$$q_r(t) = \frac{1}{2\pi i}\int_{\mathrm{Br}} e^{\omega t}\left(\frac{\omega}{\varpi} + \sqrt{\left(\frac{\omega}{\varpi}\right)^2 + 1}\right)^{-r}\frac{\mathrm{d}\omega}{\omega}\,q_0, \tag{66}$$

writing ω for $\wp$, where Br is the Bromwich contour [34].

In the continuum limit as $\chi \to 0$ ($\hbar \to 0$), and $r \to \infty$, such that $r\chi = q$ is finite and fixed, the inverse Mellin transform (66) can be evaluated by the method of steepest descent [31]. The dimensionless action

$$S(\omega) = r\ln\left(\frac{\omega}{\varpi} + \sqrt{\left(\frac{\omega}{\varpi}\right)^2 + 1}\right) - \omega t \tag{67}$$

determines the saddle points

$$\omega_\pm = \pm i\varpi\sqrt{1 - \beta^2} \tag{68}$$

from the condition

$$S'(\omega) = \frac{r/\varpi}{\sqrt{\left(\omega/\varpi\right)^2 + 1}} - t = 0.$$

Causality, $\beta < 1$, places the saddle points (68) on the *imaginary* axis.

Evaluating the action (67) at the saddle points (68) gives a purely imaginary quantity

$$S(\omega_\pm) = \pm i\left(r\cos^{-1}\beta - \varpi t\sqrt{1 - \beta^2}\right),$$

which is equivalent to (37), where $\theta = \cos^{-1}\beta$, and $0 < \theta < \pi/2$. The dispersion equation corresponds to the wave equation, (30). Because the saddle points (68) lie on the imaginary axis, the imaginary angle ϑ is transformed into a real angle θ. The dispersion equation (30) is thus equivalent to the operator correspondence (1). The leading term in the asymptotic expansion at the saddle point ω_+ is given by

$$\psi_r(\varpi t) \sim \frac{\exp\left(i\varpi t\{\sqrt{1-\beta^2} - \beta\cos^{-1}\beta\} - i\pi/4\right)}{\sqrt{\frac{1}{2}\pi\varpi t\sqrt{1-\beta^2}}},$$

which, apart from the time-dependent phase factor $e^{-i\varpi t}$, is the + component in (40).

The Ehrenfest equation (26) is the long wavelength limit of the fdd equation

$$\ddot{q}_r = \frac{1}{4}\varpi^2\left(q_{r+2} + q_{r-2} - 2q_r\right) + \varpi^2 q_r. \tag{69}$$

It is the plus sign of the mass term in (69) that converts the recurrence relation for an ordinary Bessel function (6) into that of a modified Bessel function (42). Instead of (65), we now have the subsidiary equation

$$\wp^2 = \frac{1}{2}\varpi^2\left(\cosh 2\vartheta + 1\right) = \varpi^2\cosh^2\vartheta.$$

The $\wp$-multiplied Laplace transform of q_r is

$$\wp \int_0^\infty e^{-\wp t'} q_r \, \mathrm{d}t' = \exp\left(-r\cosh^{-1}(\wp/\varpi)\right)q_0$$

for $\wp > 0$. Writing ω for $\wp$, the inverse Mellin transform now gives

$$q_r(t') = \frac{1}{2\pi i}\int_{\mathrm{Br}} e^{\omega t'}\left(\frac{\omega}{\varpi} + \sqrt{\left(\frac{\omega}{\varpi}\right)^2 - 1}\right)^{-r}\frac{\mathrm{d}\omega}{\omega}\,q_0 \tag{70}$$

for $t' > 0$.

The action

$$S(\omega) = r\ln\left(\frac{\omega}{\varpi} + \sqrt{\left(\frac{\omega}{\varpi}\right)^2 - 1}\right) - \omega t' \tag{71}$$

has a stationary value at

$$S'(\omega) = \frac{r/\varpi}{\sqrt{(\omega/\varpi)^2 - 1}} - t' = 0,$$

which places the saddle points

$$\omega_\pm = \pm\varpi\sqrt{\left(\frac{r}{\varpi t'}\right)^2 + 1}, \tag{72}$$

on the *real* axis. Evaluating the action (71) at the saddle points gives

$$S(\omega_\pm) = \pm\left(r\sinh^{-1}\left(\frac{q}{ct'}\right) - \sqrt{1 + (q/ct')^2}\,\varpi t'\right),$$

which is identical to (49). The $\pm$ signs are related to the two types of modified Bessel functions, or time reversal, rather than forward and backward propagating waves, as in the previous example.

The expression $\vartheta = \sinh^{-1}(q/ct')$ for the imaginary angle of rotation is none other than the first relation in the Lorentz transformation, (43). For any particular constant value, it moves with the group velocity (58) – and not with the phase velocity, which is the velocity that one would normally expect surfaces of constant phase to move at. Introducing this expression for the phase into the expressions for the saddle points gives

$$\omega^2 = \varpi^2\left(\sinh^2\vartheta + 1\right), \tag{73}$$

which is none other than the square of the expression for the total relativistic energy (51), in units of $\hbar$. Thus, we have a direct correspondence between the Ehrenfest equation (26) and the square of the dispersion relation, (73), through the correspondence

$$\partial_t \leftrightarrow \omega, \quad \text{and} \quad \hat{\partial}_q \leftrightarrow -\kappa, \tag{74}$$

in contrast to (1).

Again consider the Lorentz transformation from a frame O' to a frame O, in which the frame O' is moving relative to O with the velocity (58). The wave number, (53), and frequency, (50), transform like a 'two-vector' $(\kappa, i\omega/c)$, just like space and time, (q, ict), transform under the Lorentz transformation, (43). The fact that the square of their sum is invariant is simply a consequence of the hyperbolic relation, $\omega^2 - (c\kappa)^2 = \varpi^2(\cosh^2\vartheta - \sinh^2\vartheta) = \varpi^2$.

Under the correspondence (74), the Lorentz transformation

$$\begin{pmatrix} 0 \\ i\varpi/c \end{pmatrix} = \begin{pmatrix} \cosh\vartheta & i\sinh\vartheta \\ -i\sinh\vartheta & \cosh\vartheta \end{pmatrix}\begin{pmatrix} \kappa \\ i\omega/c \end{pmatrix} \tag{75}$$

is equivalent to

$$\begin{pmatrix} 0 \\ i\varpi/c \end{pmatrix} = \begin{pmatrix} \varpi^{-1}\partial_t & -i\,\chi\hat{\partial}_q \\ i\,\chi\hat{\partial}_q & \varpi^{-1}\partial_t \end{pmatrix}\begin{pmatrix} \kappa \\ i\omega/c \end{pmatrix}. \tag{76}$$

Firstly, the transformation (76) leaves the energy invariant (2). Secondly, the transformation matrix, $\mathbf{L}$, in (76) is an orthogonal matrix (*i.e.*, $\mathbf{LL}^T = \mathbf{1}$, and $\det\mathbf{L} = 1$) when the Ehrenfest equation (26) is satisfied. This is none other than the expression for the total relativistic energy. Thirdly, multiplying both sides of (76) by the transpose $\mathbf{L}^T$ gives back the operator correspondence indicated in (74).

The first equation in (76) is the exactness condition for the function of state (49), according to the equality between (56) and (57). With the aid of the dispersion relation, it can be written as

$$d_t\kappa = \left(\partial_t + v\hat{\partial}_q\right)\kappa = 0 \tag{77}$$

The wave number, frequency and, more generally, the phase, are constant on each of the characteristics. They all propagate with the group velocity, $q/t = c^2\kappa/\omega$, which are straight lines in the (q, t) plane.

The equation of energy conservation,

$$\partial_t\left(\omega a^2\right) + \hat{\partial}_q\left(c^2\kappa a^2\right) = 0, \tag{78}$$

can be written in the characteristic form

$$d_t\,a^2 = -a^2\hat{\partial}_q\,v = -\frac{a^2}{t}, \tag{79}$$

on $q/t = \partial_\kappa\omega = c^2\kappa/\omega$, so that the group, or particle, velocity is also the velocity at which the energy propagates. That κ is a constant on each characteristic ordinarily means that the medium is uniform, while constant frequency on each characteristic means that the process is stationary [17]. In the present context the fact that both are constants on each characteristic is a consequence of the condition that the frame be inertial. The term, ωa^2, can again be thought of as the energy density, but now $c^2\kappa a^2$ is the corresponding energy flux. This would correspond to the energy flux defined in (21) if v were the group velocity $c^2\kappa/\omega$. But, the group velocity there was given as a periodic

function of the wave number, (18). It is only when the mass is introduced that the group velocity becomes proportional to the wave number at small values of the latter, as can be gleaned from (39). To first order in $\kappa \chi$, $v \approx \kappa \hbar / m$ since to this order $\omega \approx \varpi$. The group velocity, $v = c^2 \kappa / \omega$, is still the ratio of the energy flux to the energy density. The amplitude equation may be integrated along the characteristic, giving (23). Consequently, the intensity a^2 propagates with the group velocity, and decays because of the divergence of the group lines with a separation increasing in proportion to t [17].

However, the fact that the frame is inertial means there is the constraint of constant relative velocity. According to (44), this requires the phase ϑ to also be constant. If the usual definitions of frequency and wave number were used, it would equate the particle velocity with the phase velocity, and not with the group velocity!

The constraint that the frame be inertial is incorporated into the Lorentz transformation and, equivalently, into (76). The second condition to be met is

$$\partial_t \omega + c^2 \hat{\partial}_q \kappa = \varpi^2, \tag{80}$$

which is comparable with energy conservation (78), provided the intensity a^2 of the wave train satisfies

$$d_t a^2 = -\frac{\varpi^2}{\omega} a^2, \tag{81}$$

on $q/t = c^2 \kappa / \omega$. Equation (81) is easily integrated along the characteristic to give

$$a^2(t) = a_0^2 \exp\left(-\varpi^2 \frac{t}{\omega}\right) = a_0^2 \exp\left(-\varpi t'\right), \tag{82}$$

because the frequency is a constant along the characteristic, and a_0 is determined by the initial condition. Under the constraint of constant group velocity, it is the mass that causes the a^2 to vary in time; not as t^{-1}, but exponentially. We will now identify (82) as the integrating factor for constant phase

$$\vartheta(q, t) = \vartheta_0, \tag{83}$$

cf. (86) and (87), provided it moves with the group velocity.

Under phase reversal, $\vartheta \rightarrow -\vartheta$, the inverse of (82) will be an integrating factor, which is derived by switching the minus sign between

$$\partial_t \vartheta = -\omega \quad \text{and} \quad \hat{\partial}_q \vartheta = \kappa. \tag{83a}$$

The change in sign leaves the determinant of the Lorentz transformation invariant. One minus sign is required in order to obtain the correct consistency condition (17). And switching signs in (74) requires switching them in (83a). This is the same symmetry in past and future manifested by the modified Bessel functions of first and second kinds, (59) and (60).

It follows from expression (44) that the condition for a frame to be inertial is (83), for any particular value ϑ_0. Consequently, surfaces of constant phase move with the velocity

$$-\frac{\partial_t \vartheta}{\hat{\partial}_q \vartheta} = \frac{\omega}{\kappa} = c \coth \vartheta. \tag{84}$$

However, this violates causality, as well as being at odds with the consistency condition. The characteristic form (77) shows that any particular value κ_0 will be found at points $q = v(\kappa_0)t$. In other words, an observer moving with velocity $v(\kappa_0)$ will always see waves with wave number κ_0. In a dispersive medium this is not the same as the velocity at which the phase moves, (84). For this reason the phase velocity is commonly believed to have "no physical significance" [1], so that "de Broglie wave phases cannot be used for the transmission of signals" [2].

There are, however, at least two arguments against this: (i) the condition for an inertial frame implies (83), and (ii) it is precisely (84) that yields the relativistic Hamilton-Jacobi equation. Upon squaring both sides and using the definition (53) there results [16, p. 28]

$$\left(\partial_t \vartheta\right)^2 - c^2 \left(\hat{\partial}_q \vartheta\right)^2 = \varpi^2. \tag{85}$$

Hence, the relativistic Hamilton-Jacobi equation (85) is a direct consequence of the fact that the phase travels at the superluminal phase velocity (84).

The definition of the phase velocity as (84), on the condition that the frame be inertial (83), leads to the seemingly inescapable conclusion that the particle travels faster than light. The origin of what makes a^2 decay in time should be immaterial, regardless of whether it is due to the divergence $\hat{\partial}_q v$ of the group lines, or to the presence of mass. Equating the right hand sides of the characteristic equations (79) and (81) for the intensity, which behaves in every respect as an energy-like quantity [17, p. 389], gives

$\hat{\partial}_q v = \varpi^2/\omega$, or

$$\hat{\partial}_q \vartheta = \frac{\omega}{c}. \tag{86}$$

Moreover, due to the constraint that the group velocity is constant, $d_t v = 0$, we also have $\partial_t v/v = -\varpi^2/\omega$, or

$$\partial_t \vartheta = -\kappa c. \tag{87}$$

In the absence of dispersion, *i.e.*, zero rest mass, these definitions of frequency and wave number are indistinguishable from the ordinary ones. The interchange of the roles of ω and κ means that the consistency condition (17) will be replaced by

$$\partial_t \omega + c^2 \hat{\partial}_q \kappa = 0,$$

which coincides with (80) in the zero mass limit. In this limit, (86) and (87) can be substituted into (17) to obtain the wave equation.

Notwithstanding the presence of a finite rest mass, $d\vartheta = 0$ can be made an exact differential through an appropriate integrating factor, which always exist for a Pfaffian of two independent variables. The integrating factor turns out to be (82), which, we recall, is the solution to the second equation in (76). For then we have

$$d\tilde{\vartheta} = a^2(t) \left(\frac{\omega}{c} d q - c\kappa\, dt \right) = 0,$$

whose condition of exactness is precisely (80). And what was the exactness condition for the ordinary phase, (17), now provides the expression for the group velocity

$$-\frac{\partial_t \tilde{\vartheta}}{\hat{\partial}_q \tilde{\vartheta}} = c \tanh \tilde{\vartheta}. \tag{88}$$

Thus, the condition for an inertial frame of reference, $d\vartheta = 0$, which always coincides with a solution of $d\tilde{\vartheta} = 0$, fixes the relative velocity at the group velocity, (88).

The Pfaffian equation $d\vartheta = 0$ defines a definite direction at each point in the (q, t) plane. The tangent to the curve at any point has the direction specified by $d\vartheta = 0$ whose solution

$$\tilde{\vartheta}(q, t) = \frac{1}{c} \int_{q_0}^q \omega\, e^{-\varpi^2 t/\omega}\, d q = \tilde{\vartheta}_0 \tag{89}$$

defines a one-parameter family of curves in the (q, t) plane, parameterized by the constant $\tilde{\vartheta}_0$. From this we conclude that *the mode in which an energy-like quantity decays in time under the constraint that the frame be inertial is the integrating factor for the phase which propagates at the group velocity.* Phase reversal $\tilde{\vartheta} \to -\tilde{\vartheta}$ requires time reversal, $t \to -t$, so that we can immediately obtain the time reversal solution from (89). Under phase reversal, the condition of integrability

$$\partial_t \omega + c^2 \hat{\partial}_q \kappa = -\varpi^2, \tag{90}$$

is obtained by switching the signs in the operator correspondence (74).

Squaring both sides (88) gives the eikonal equation,

$$\left(\partial_t \tilde{\vartheta} \right)^2 - v^2 \left(\hat{\partial}_q \tilde{\vartheta} \right)^2 = 0, \tag{91}$$

showing that $\tilde{\vartheta}$ propagates with the velocity $\pm v$, where $v = c^2 \kappa/\omega$ is the group velocity. The condition that the system is inertial is analogous to the condition that the fluid is homogeneous. In other words, the wave front moves with the particle velocity. Rather, had we adhered to the usual definitions of frequency and wave number, and squared both sides of (84), the corresponding phase would have been found to travel at a superluminal velocity.

The eikonal equation (91) can be cast in the form of the Hamilton-Jacobi equation

$$\left(\partial_t \tilde{\vartheta} \right)^2 - c^2 \left(\hat{\partial}_q \tilde{\vartheta} \right)^2 = -\varpi^2, \tag{92}$$

where definition (86) has been used. But, precisely because of the definitions of frequency and wave number, (86) and (87), the Hamilton-Jacobi equation (92) is equivalent to the dispersion relation (2), and hence to (85). The negative sign on the right-hand side of (92) acknowledges the fact that the roles of q and t have been interchanged. Its advantage over (85) is that it avoids having to introduce the superluminal phase velocity (84). In the realm of geometric optics, which is equivalent to the domain of validity of the Hamilton-Jacobi equation, the wave nature of the motion has disappeared, and particle trajectories correspond to light rays which propagate in an orthogonal direction to surfaces of constant phase. There is only a single velocity, that of the particle. Hence, if we were to use the standard definitions of frequency

and wave number, (84) would lead to the conclusion that the particle's velocity would be greater than that of light.

Notwithstanding the negative sign in (92), it reduces to the nonrelativistic Hamilton-Jacobi equation in the limit as $c \to \infty$. For both equations are equivalent to the dispersion equation (2). Subtracting off the rest frequency, $\omega' = \omega - \varpi$, and substituting it into the dispersion relation results in

$$\left(\frac{\omega'}{c}\right)^2 + 2m\omega'/\hbar = \kappa^2.$$

In the limit as $c \to \infty$, this equation goes over into the dispersion equation for a nonrelativistic free-particle.

6. Phase Correlations

The phase relation for (63) between neighboring sites along the string is

$$q_{r+1} = e^{-i\cos^{-1}\beta} q_r.$$

If β is a little less than 1, the change in phase from one particle to the next along the string is small, whereas as β approaches zero, the phase approaches $\pi/2$. But this is just the generalized Feynman prescription of "loading each turn through θ by $e^{i\theta}$"[12], for any value of θ in $0 \le \theta \le \pi/2$. This interval is a consequence of employing the symmetric difference quotient in the string equation (63), which reduces the Brillouin zone by half its nonrelativistic size. Had the product of the forward and backward difference quotients been used, as in the fdd Schrödinger equation (33), the Brillouin zone would have been doubled. As $\theta \to \pi$, the nearest neighbors have the same amplitudes and opposite phases, thereby establishing a standing wave, known as the π-mode [22, p. 66].

In contrast, the direct correlation for (69), between neighboring sites along the string, is

$$q_{r+1} = e^{-\tanh\beta} q_r = \sqrt{\frac{1-\beta}{1+\beta}}\, q_r. \tag{93}$$

The amplification factor is the radial Doppler effect. The amplitude of a neighboring site to which the disturbance has propagated will be modified by the direction in which it is travelling. It will either be 'red' shifted, $\beta > 0$ if the disturbance is travelling away from it, or 'blue' shifted, $\beta < 0$, if the disturbance is approaching it. The greatest change occurs as the disturbance approaches the speed of light. The total amplitude change after the disturbance has travelled a distance of $r\chi$ is $k^r = e^{-\vartheta r}$, which is the well-known multiplication rule of Bondi's k-calculus [35].

Solving (93) for β we get

$$\beta = \frac{1 - (q_{r+1}/q_r)^2}{(q_{r+1}/q_r)^2 + 1}. \tag{94}$$

Realizing that the spacing between $r + 1$ and r is arbitrary, we can further divide this interval up into sub-intervals and write (94) as

$$\beta = \frac{1 - (q_{r+1}/q_{r+\alpha})^2(q_{r+\alpha}/q_r)^2}{(q_{r+1}/q_{r+\alpha})^2(q_{r+\alpha}/q_r)^2 + 1} \tag{95}$$

for any $\alpha \in (0, 1)$. If the particle propagates with a relative velocity v' in the first sub-interval, and a relative velocity v'' in the second sub-interval, then (95) is none other than

$$\beta = \frac{\beta' + \beta''}{1 + \beta'\beta''},$$

which is the relativistic law for the addition of velocities.

Acknowledgement

[1] M. Born, Atomic Physics; Dover, New York 1969, p. 90.

[2] P. Bergmann, Introduction to the Theory of Relativity; Dover, New York, 1976, p. 145.

[3] M. Laue, Ann. Physik [4] **18**, 523 (1905).

[4] A. Sommerfeld, Ann. Physik [4] **4**, 177 (1914); translated in [15], Chapt. 2.

[5] W. Feller, SIAM **14**, 864 (1966); Introduction to Probability and Its Applications; Wiley, New York 1971, vol. II, p. 58–61.

[6] B. H. Lavenda, Nuovo Cimento **113B**, in press.

[7] R. D. Richtmyer and K. W. Morton, Difference Methods for Initial-Value Problems; 2nd ed., Interscience, New York, 1967, p. 12.

[8] J. J. Sakurai, Advanced Quantum Mechanics; Addison-Wesley, Reading MA, 1967, p. 169.

[9] F. John, Partial Differential Equations; 3rd ed., Springer-Verlag, New York 1978, p. 7.

[10] A. Einstein, Jahrbuch der Radioaktivität **4**, 412 (1912).

[11] L. S. Schulman, in Path Summation: Achievements and Goals, S. O. Lundquist, A. Ranfagni, V. Sayakanit, and L. S. Schulman, eds.; World Scientific, Singapore, 1988, pp. 3 - 46.

[12] S. S. Schweber, Reviews of Modern Physics **58**, 449 (1986).

[13] V. Thomée, in Handbook of Numerical Analysis; P. G. Ciarlet and J. L. Lions, eds.; North-Holland, Amsterdam, 1990, vol. I, pp. 5 - 196.

[14] P. Debye, in The Collected Papers of Peter J. W. Debye; Ox Bow Press, Woodbridge CT 1988, p. 583.

[15] L. Brillouin, Wave Propagation and Group Velocity; Academic Press, New York 1960, p. 122.

[16] L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields; Pergamon, Oxford, 1975, 4th ed., pp. 84–85.

[17] G. B. Whitham, Linear and Nonlinear Waves; Wiley, New York, 1974, pp. 376 - 380.

[18] T. A. Welton, Phys. Rev. **74**, 1157 (1948).

[19] H. A. Bethe, Phys. Rev. 72, 339 (1947).

[20] L. Brillouin, Wave Propagation in Periodic Structures; 2nd ed., Dover Pub., New York 1953, p. 4.

[21] E. R. Fitzgerald, Particle Waves and Deformation in Crystalline Solids; Wiley, New York 1966, p. 21.

[22] R.A. Smith, Wave Mechanics of Crystalline Solids; Chapman & Hall, London 1961, p. 122.

[23] P. Ehrenfest, Ann. Physik [4] **33**, 1571 (1910).

[24] W. Voigt, Ann. Physik [3] **68**, 598 (1899); [4] **4**, 203 (1901).

[25] B. Gaveau, T. Jacobson, M. Kac, and L. S. Schulman, Phys. Rev. Lett. **53**, 419 (1984).

[26] Kac, M., Rocky Mountain J. Math. **4**, 497 (1974).

[27] B. de Finetti, Theory of Probability; Wiley, Chichester 1974, vol. 2, p. 80.

[28] N. Bleistein and R. A. Handelsman, Asymptotic Expansion of Integrals; Dover, New York 1986, pp. 272.

[29] W. Heisenberg, Zeit. Phys. **65**, 4 (1930); translated in A. I. Miller, Early Quantum Electrodynamics: A source book; Cambridge U. P., Cambridge 1994, p. 121.

[30] B. H. Lavenda, Z. Naturforsch. **55a**, 745 (2000).

[31] H. Jeffreys and B. Jeffreys, Methods of Mathematical Physics; Cambridge U. P., Cambridge 1956, p. 597 - 599.

[32] L. de Broglie, Nonlinear Wave Mechanics: A Causal Interpretation; Elsevier, Amsterdam 1956, p. 5.

[33] R. P. Feynman, Quantum Electrodynamics; Benjamin, Reading MA 1962, p. 68.

[34] N. W. McLachlan, Complex Variable and the Operational Calculus with Technical Applications; Cambridge U. P., Cambridge 1963, 2nd ed., p. 135.

[35] H. Bondi, Relativity and Common Sense; Dover, New York 1964, pp. 88 - 92.